

THE DIRICHLET PROBLEM FOR FULLY NONLINEAR ELLIPTIC EQUATIONS ON RIEMANNIAN MANIFOLDS

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ABSTRACT. We study a class of fully nonlinear elliptic equations on Riemannian manifolds and solve the Dirichlet problem in a domain with no geometric restrictions to the boundary under essentially optimal structure conditions. It includes a new (and optimal) result in the Euclidean case (see Theorem 1.1). We introduce some new methods in deriving *a priori* C^2 estimates, which can be used to treat other types of fully nonlinear elliptic and parabolic equations on real or complex manifolds.

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1. INTRODUCTION

In a seminal paper [1] Caffarelli, Nirenberg and Spruck established fundamental existence results on the Dirichlet problem in \mathbb{R}^n for fully nonlinear elliptic equations of the form

$$(1.1) \quad F(\nabla^2 u) \equiv f(\lambda(\nabla^2 u)) = \psi \text{ in } \Omega \subset \mathbb{R}^n$$

where $\lambda(\nabla^2 u) = (\lambda_1, \dots, \lambda_n)$ are the eigenvalues of $\nabla^2 u$, Hessian of $u \in C^2(\Omega)$, and f is a smooth symmetric function of n variables defined in a symmetric open and convex cone $\Gamma \subset \mathbb{R}^n$ with vertex at the origin and boundary $\partial\Gamma \neq \emptyset$, and $\Gamma_n \subseteq \Gamma$ where

$$(1.2) \quad \Gamma_n \equiv \{\lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0\}.$$

Since [1] equations of form (1.1) and its variations have received extensive study, and the fundamental structure conditions introduced in [1] have become standard in the literature of fully nonlinear elliptic and parabolic equations. These include

$$(1.3) \quad f_i = f_{\lambda_i} \equiv \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma, \quad 1 \leq i \leq n,$$

$$(1.4) \quad f \text{ is a concave function in } \Gamma$$

and

$$(1.5) \quad \delta_{\psi, f} \equiv \inf_M \psi - \sup_{\partial\Gamma} f > 0, \text{ where } \sup_{\partial\Gamma} f \equiv \sup_{\lambda_0 \in \partial\Gamma} \limsup_{\lambda \rightarrow \lambda_0} f(\lambda).$$

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In this paper we wish to solve the Dirichlet problem for equation (1.1) under these basic assumptions which are essentially optimal.

Theorem 1.1. *Let Ω be a bounded smooth domain in \mathbb{R}^n ($n \geq 2$), $\psi \in C^\infty(\bar{\Omega})$ and $\varphi \in C^\infty(\partial\Omega)$. Suppose that (1.3)-(1.5) hold and that there exists an admissible subsolution $\underline{u} \in C^2(\bar{\Omega})$ satisfying $\underline{u} = \varphi$ on $\partial\Omega$ and*

$$(1.6) \quad f(\lambda(\nabla^2 \underline{u})) \geq \psi \text{ in } \bar{\Omega}.$$

There exists a unique admissible solution $u \in C^\infty(\bar{\Omega})$ of equation (1.1) with $u|_{\partial\Omega} = \varphi$.

A function $u \in C^2(\Omega)$ is called *admissible* if $\lambda(\nabla^2 u) \in \Gamma$. It was shown in [1] that condition (1.3) implies that equation (1.1) is elliptic for admissible solutions while by condition (1.4) the function F defined by $F(A) = f(\lambda[A])$ is concave for $A \in \mathcal{S}^{n \times n}$ with $\lambda[A] \in \Gamma$, where $\mathcal{S}^{n \times n}$ is the set of n by n symmetric matrices. Condition (1.5) prevents equation (1.1) from being degenerate. It also ensures that equation (1.1) becomes uniformly elliptic once *a priori* bounds for $C^2(\bar{\Omega})$ norm of admissible solutions are established so that one can apply Evans-Krylov theorem, which heavily relies on the concavity assumption (1.4), to obtain $C^{2,\alpha}$ estimates. From this point of view conditions (1.3)-(1.5) are all crucial to the classical solvability of equation (1.1).

Theorem 1.1 was proved by Caffarelli, Nirenberg and Spruck [1] under additional conditions on f in a domain Ω satisfying the geometric condition that there exists $R > 0$ such that

$$(1.7) \quad (\kappa_1, \dots, \kappa_{n-1}, R) \in \Gamma \text{ on } \partial M$$

where $(\kappa_1, \dots, \kappa_{n-1})$ are the principal curvatures of ∂M . The theory and techniques developed in [1] have had huge impact to the study of fully nonlinear elliptic and parabolic equations and applications. In [9] Trudinger was able to improve their results, proving Theorem 1.1 under (1.3)-(1.5), (1.7) and the assumption that for every $C > 0$ and compact set K in Γ there is a number $R = R(C, K)$ such that

$$(1.8) \quad f(R\lambda) \geq C \text{ for all } \lambda \in K.$$

Trudinger [9] introduced a new idea in deriving second order boundary estimates, which I found very useful in my previous work and will be used in this paper; see Section 3.

We remark that in Theorem 1.1 there is no geometric restrictions to the boundary of Ω . So it is necessary to assume the existence of a subsolution, or otherwise the Dirichlet problem may not always be solvable. It was shown in [1] that using (1.7) and (1.8) one can construct admissible *strict* subsolutions. Motivated by work in [7] and [6] on Monge-Ampère equations and their geometric applications, the author made attempts in [2], [3] and more recently in [4] (see also [5]) to study equation (1.1) in general domains as well as on Riemannian manifolds, but always needed extra hypotheses to overcome difficulties arising from different technical issues. In this

paper we introduce some new ideas which enable us to derive the estimates needed in the proof of Theorem 1.1 under the basic assumptions (1.3)-(1.6).

Our second result in this paper is the following extension of Theorem 1.1 to general Riemannian manifolds under an additional technical condition on f .

Theorem 1.2. *Let M^n be a compact Riemannian manifold of dimension $n \geq 2$ with smooth boundary ∂M . Suppose in addition that f satisfies*

$$(1.9) \quad \sum f_i(\lambda) \lambda_i \geq 0 \quad \text{in } \Gamma \cap \{\inf_M \psi \leq f \leq \sup_M \psi\}.$$

Theorem 1.1 still holds for $\Omega = M$.

The key to solving the Dirichlet problem for equation (1.1) is to establish *a priori* C^2 estimates for admissible solutions. For Theorem 1.1 a major challenge comes from second order boundary estimates for a domain with arbitrary boundary shape. Technically the proof of Theorem 1.2 is however much more complicated as one also encounters substantial difficulties in deriving global estimates for both gradient and second derivatives due to the presence of curvature. Our primary goal in this paper is to seek methods to overcome these difficulties. We shall present our results for the more general Dirichlet problem

$$(1.10) \quad \begin{cases} f(\lambda(\nabla^2 u + \chi)) = \psi & \text{in } M, \\ u = \varphi & \text{on } \partial M \end{cases}$$

where χ is a smooth $(0, 2)$ tensor on \bar{M} . Throughout the paper we assume that (M^n, g) is a compact Riemannian manifold of dimension $n \geq 2$ with smooth boundary ∂M , $\bar{M} = M \cup \partial M$, and that $\psi \in C^\infty(\bar{M})$ and $\varphi \in C^\infty(\partial M)$. The inclusion of χ in (1.10) is natural and important in applications. Accordingly we call a function $u \in C^2(M)$ *admissible* if $\lambda(\nabla^2 u + \chi) \in \Gamma$.

Theorem 1.3. *Let $u \in C^4(M) \cap C^2(\bar{M})$ be an admissible solution of the Dirichlet problem (1.10). Assume (1.3)-(1.5) and that there exists an admissible subsolution $\underline{u} \in C^2(\bar{M})$:*

$$(1.11) \quad \begin{cases} f(\lambda[\nabla^2 \underline{u} + \chi]) \geq \psi & \text{in } \bar{M}, \\ \underline{u} = \varphi & \text{on } \partial M. \end{cases}$$

There exists C depending on $|u|_{C^1(\bar{M})}$, $|\underline{u}|_{C^2(\bar{M})}$ and other known data such that

$$(1.12) \quad \max_{\bar{M}} |\nabla^2 u| \leq C \left(1 + \max_{\partial M} |\nabla^2 u| \right).$$

Suppose in addition that

$$(1.13) \quad \sum f_i(\lambda) \lambda_i \geq -K_0 \left(1 + \sum f_i \right) \quad \text{in } \Gamma \cap \{\inf_M \psi \leq f \leq \sup_M \psi\}.$$

for some $K_0 \geq 0$. Then

$$(1.14) \quad \max_{\partial M} |\nabla^2 u| \leq C.$$

Furthermore, if $K_0 = 0$ and there exists a function $w \in C^2(\bar{M})$ with $\nabla^2 w \geq \chi$ then

$$(1.15) \quad |u|_{C^1(\bar{M})} \leq C.$$

Equation (1.10) on closed Riemannian manifolds was first studied by Y.-Y. Li [8] for $\chi = g$, followed by Urbas [10] while the Dirichlet problem was considered more recently by the author [4] where the second order estimates (1.12) and (1.14) were derived under an additional condition (see description below).

The concavity condition (1.4) is extremely important in the theory of fully nonlinear equations. It is a cornerstone to Evans-Krylov theorem, and fundamental to second order estimates as well. Nevertheless, using ideas introduced in [4] we are able to weaken the assumption. Let us first recall some notation and results from [4].

For $\sigma > \sup_{\partial\Gamma} f$, define $\Gamma^\sigma = \{\lambda \in \Gamma : f(\lambda) > \sigma\}$ and suppose $\Gamma^\sigma \neq \emptyset$. By (1.3) and (1.4) the level set $\partial\Gamma^\sigma = \{\lambda \in \Gamma : f(\lambda) = \sigma\}$ is a smooth convex hypersurface in \mathbb{R}^n . Define

$$S_\mu^\sigma = \{\lambda \in \partial\Gamma^\sigma : \mu \in T_\lambda \partial\Gamma^\sigma\} \text{ for } \mu \in \Gamma \setminus \Gamma^\sigma$$

and $\mathcal{C}_\sigma^+ = V_\sigma \cup \Gamma^\sigma$ which we call the *tangent cone at infinity* to Γ^σ , where $T_\lambda \partial\Gamma^\sigma$ denotes the tangent plane of $\partial\Gamma^\sigma$ at λ and

$$V_\sigma = \{\mu \in \Gamma \setminus \Gamma^\sigma : S_\mu^\sigma \text{ is nonempty and compact}\}.$$

It is shown in [4] that \mathcal{C}_σ^+ is an open convex subset of Γ , and that (1.12) and (1.14) hold provided in addition that $\partial\Gamma^\sigma \cap \partial\mathcal{C}_\sigma^+ = \emptyset$, or equivalently

$$(1.16) \quad \partial\Gamma^\sigma \subset \mathcal{C}_\sigma^+, \quad \forall \sigma \in \left[\inf_M \psi, \sup_M \psi \right].$$

We note that if in place of (1.4) we assume that

$$(1.17) \quad f \text{ is concave in } \Gamma \setminus B_{R_0}(0)$$

for some $R_0 > 0$ where $B_{R_0}(0)$ is the ball of radius R_0 centered at the origin, then \mathcal{C}_σ^+ is still well defined (see Section 5).

Theorem 1.4. *Theorem 1.3 still holds with (1.4) replaced by (1.17) and (1.16) .*

We should remark that Theorem 1.4 is, however, not an extension of Theorem 1.3, and it would be interesting to weaken (1.16) in Theorem 1.4 to

$$(1.18) \quad \partial\Gamma^\sigma \subset \overline{\mathcal{C}_\sigma^+}, \quad \forall \sigma \in \left[\inf_M \psi, \sup_M \psi \right]$$

which is implied by (1.4). Another interesting question would be whether one can prove Evans-Krylov theorem with (1.4) replaced by (1.17) and (1.18) or (1.16). During a conversation in August 2012, this question was also raised to the author by Fanghua Lin to whom we wish to express our gratitude.

In Theorem 1.3 it would be desirable to remove the assumption $\nabla^2 w \geq \chi$ which is only needed in order to derive the gradient estimate

$$(1.19) \quad \max_M |\nabla u| \leq C(1 + \max_{\partial M} |\nabla u|)$$

where C depends on $|u|_{C^0(\bar{M})}$. The condition is obviously satisfied if $\chi = 0$ or there is a strictly convex function on \bar{M} . Unlike in \mathbb{R}^n , deriving gradient estimates on Riemannian manifolds has been rather difficult for reasons such as the presence of curvature and lack of globally defined functions with desired properties, e.g. convex functions. There are other conditions which have been used in gradient estimates; see [8], [10], [4] and Section 4.

As a consequence of our *a priori* estimates, we may prove the following existence result of which Theorem 1.2 is clearly a special case.

Theorem 1.5. *Under conditions (1.3)-(1.5), (1.13) and (1.11) there exists a unique admissible solution $u \in C^\infty(\bar{M})$ of the Dirichlet problem (1.10) provided that any one of the following assumptions is satisfied: (i) $\Gamma = \Gamma_n$; (ii) (M, g) has nonnegative sectional curvature; (iii) there is $\delta_0 > 0$ such that*

$$(1.20) \quad f_j \geq \delta_0 \sum f_i \text{ if } \lambda_j < 0, \text{ on } \partial\Gamma^\sigma \forall \sigma > \sup_{\partial\Gamma} f;$$

$$(iv) K_0 = 0 \text{ in (1.13) and } \nabla^2 w \geq \chi \text{ for some function } w \in C^2(\bar{M}).$$

The rest of this paper is organized as follows. The following three sections contain the proof of Theorem 1.3; we establish (1.12) in Section 2, (1.14) in Section 3 and (1.15) in Section 4 where the gradient estimate (1.19) is also derived for the other cases in Theorem 1.5. Finally in Section 5 we prove Theorem 1.4.

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2. GLOBAL ESTIMATES FOR SECOND DERIVATIVES

In this section we derive the global second order estimates (1.12) under hypotheses (1.3)-(1.5) and (1.11). The beginning part of the proof will be essentially same as in [4]; we give an outline of this part for completeness and reader's convenience, followed by our new ideas which will also be critical in the following sections. In the proof we shall follow [4] for notations, and keep track explicitly the dependence on $|\nabla u|_{C^0(\bar{M})}$.

As in [4] we consider

$$W = \max_{x \in \bar{M}} \max_{\xi \in T_x M^n, |\xi|=1} (\nabla_{\xi\xi} u + \chi(\xi, \xi)) e^\eta$$

where η is a function to be determined. Suppose W is achieved at an interior point $x_0 \in M$ for some unit vector $\xi \in T_{x_0} M^n$ and choose smooth orthonormal local frames e_1, \dots, e_n about x_0 such that $e_1 = \xi$, $\nabla_i e_j = 0$ and $\{\nabla_{ij} u + \chi_{ij}\}$ is diagonal at x_0 .

Write $U = \nabla^2 u + \chi$ and $U_{ij} = \nabla_{ij} u + \chi_{ij}$. At the point x_0 we have for $1 \leq i \leq n$,

$$(2.1) \quad \frac{\nabla_i U_{11}}{U_{11}} + \nabla_i \eta = 0,$$

$$(2.2) \quad \frac{\nabla_{ii}U_{11}}{U_{11}} - \left(\frac{\nabla_i U_{11}}{U_{11}} \right)^2 + \nabla_{ii}\eta \leq 0.$$

Next, write equation (1.1) in the form

$$(2.3) \quad F(U) := f(\lambda(U)) = \psi$$

and $F^{ij} = F^{ij}(U) = \partial F / \partial U_{ij}$. Differentiating equation (2.3) twice yields

$$(2.4) \quad F^{ij} \nabla_k U_{ij} = \nabla_k \psi, \text{ for all } k,$$

$$(2.5) \quad F^{ij} \nabla_{11} U_{ij} + \sum F^{ij,kl} \nabla_1 U_{ij} \nabla_1 U_{kl} = \nabla_{11} \psi.$$

Recall the formula

$$(2.6) \quad \begin{aligned} \nabla_{ijkl}v - \nabla_{klij}v &= R_{ljk}^m \nabla_{im}v + \nabla_i R_{ljk}^m \nabla_m v + R_{lik}^m \nabla_{jm}v \\ &+ R_{jik}^m \nabla_{lm}v + R_{jil}^m \nabla_{km}v + \nabla_k R_{jil}^m \nabla_m v. \end{aligned}$$

Therefore,

$$(2.7) \quad F^{ii} \nabla_{ii} U_{11} \geq F^{ii} \nabla_{11} U_{ii} - C(|\nabla u| + U_{11}) \sum F^{ii}.$$

In this proof the constant C , which may change from line to line, will be independent of the gradient bound $|\nabla u|_{C^0(\bar{M})}$. From (2.2), (2.5) and (2.7) we derive

$$(2.8) \quad U_{11} F^{ii} \nabla_{ii} \eta \leq E - \nabla_{11} \psi + C(|\nabla u| + U_{11}) \sum F^{ii}$$

where

$$E \equiv F^{ij,kl} \nabla_1 U_{ij} \nabla_1 U_{kl} + \frac{1}{U_{11}} F^{ii} (\nabla_i U_{11})^2.$$

Let

$$J = \{i : 3U_{ii} \leq -U_{11}\}, \quad K = \{i > 1 : 3U_{ii} > -U_{11}\}.$$

We may modify the estimate for E in [4] using the formula

$$(2.9) \quad \nabla_{ijk}v - \nabla_{jik}v = R_{kij}^l \nabla_l v$$

to obtain

$$(2.10) \quad E \leq U_{11} \sum_{i \in J} F^{ii} (\nabla_i \eta)^2 + C U_{11} F^{11} \sum_{i \notin J} (\nabla_i \eta)^2 + \frac{C(1 + |\nabla u|^2)}{U_{11}} \sum F^{ii}.$$

As in [4] we choose the function η of the form

$$\eta = \phi(|\nabla u|^2) + a(\underline{u} - u)$$

where ϕ is a positive function, $\phi' > 0$, and a is a positive constant. Then

$$\nabla_i \eta = 2\phi' (U_{ii} \nabla_i u - \chi_{ik} \nabla_k u) + a \nabla_i (\underline{u} - u),$$

$$\nabla_{ii} \eta = 2\phi' (\nabla_{ik} u \nabla_{ik} u + \nabla_k u \nabla_{ik} u) + 2\phi'' (\nabla_k u \nabla_{ik} u)^2 + a \nabla_{ii} (\underline{u} - u).$$

Therefore,

$$(2.11) \quad \sum_{i \in J} F^{ii} (\nabla_i \eta)^2 \leq 8(\phi')^2 \sum_{i \in J} F^{ii} (\nabla_k u \nabla_{ik} u)^2 + C(1 + |\nabla u|^2) a^2 \sum_{i \in J} F^{ii},$$

$$(2.12) \quad \sum_{i \notin J} (\nabla_i \eta)^2 \leq C(1 + |\nabla u|^2)(a^2 + (\phi')^2 U_{11}^2)$$

and by (2.4),

$$(2.13) \quad \begin{aligned} F^{ii} \nabla_{ii} \eta &\geq \phi' F^{ii} U_{ii}^2 + 2\phi'' F^{ii} (\nabla_k u \nabla_{ik} u)^2 + a F^{ii} \nabla_{ii} (\underline{u} - u) \\ &\quad - C\phi' |\nabla u|^2 \sum F^{ii} - C\phi' |\nabla u|. \end{aligned}$$

Let $b_1 = 1 + \max_{\bar{M}} |\nabla u|^2$, $b = \gamma/b_1^2$ where $\gamma \in (0, 1/8]$ will be chosen small enough, and $\phi(t) = -\log(1 - bt^2)$. We have

$$\phi' = \frac{2bt}{1 - bt^2}, \quad \phi'' = \frac{2b + 2b^2 t^2}{(1 - bt^2)^2}$$

and therefore

$$\phi'' - 4(\phi')^2 = \frac{2b - 14b^2 t^2}{(1 - bt^2)^2} > 0, \quad \forall 1 \leq t \leq b_1.$$

Combining (2.8) and (2.10)-(2.13) we obtain

$$(2.14) \quad \begin{aligned} \phi' F^{ii} U_{ii}^2 + a F^{ii} \nabla_{ii} (\underline{u} - u) &\leq C t a^2 \sum_{i \in J} F^{ii} + C t (a^2 + (\phi')^2 U_{11}^2) F^{11} \\ &\quad - \frac{\nabla_{11} \psi}{U_{11}} + C t \left(1 + \sum F^{ii} \right) \end{aligned}$$

where $t = 1 + |\nabla u|^2$ and C is independent of $|\nabla u|_{C^0(\bar{M})}$.

So far we have essentially followed [4] except the choice of function ϕ and the explicit dependence on $|\nabla u|_{C^0(\bar{M})}$. Our new ideas in the proof are present below.

Write $\mu(x) = \lambda(\nabla^2 \underline{u}(x) + \chi(x))$ and note that $\{\mu(x) : x \in \bar{M}\}$ is a compact subset of Γ . There exist uniform constants $\beta, \delta \in (0, \frac{1}{2\sqrt{n}})$ such that

$$(2.15) \quad \nu_{\mu(x)} - 2\beta \mathbf{1} \in \Gamma_n, \quad \forall x \in \bar{M}$$

where $\nu_\lambda := Df(\lambda)/|Df(\lambda)|$ is the unit normal vector to the level hypersurface $\partial\Gamma^{f(\lambda)}$ for $\lambda \in \Gamma$ and, by the smoothness of f and $\partial\Gamma^{f(\mu(x))}$,

$$\inf_{x \in \bar{M}} \text{dist}(\partial B_\delta^\beta(\mu(x)), \partial\Gamma^{f(\mu(x))}) > 0$$

where $\partial B_\delta^\beta(\mu)$ denotes the spherical cap

$$\partial B_\delta^\beta(\mu) = \{\zeta \in \partial B_\delta(\mu) : |\nu_\mu \cdot (\zeta - \mu)|/\delta \geq \beta \sqrt{1 - \beta^2/4}\}.$$

Therefore,

$$(2.16) \quad \theta \equiv \inf_{x \in \bar{M}} \inf_{\zeta \in \partial B_\delta^\beta(\mu(x))} (f(\zeta) - f(\mu(x))) > 0.$$

Let $\tilde{\mu} = \mu(x_0)$ and $\tilde{\lambda} = \lambda(U(x_0))$.

Lemma 2.1. *Suppose that $|\nu_{\tilde{\mu}} - \nu_{\tilde{\lambda}}| \geq \beta$. Then for some uniform constant $\varepsilon > 0$,*

$$(2.17) \quad \sum f_i(\tilde{\lambda})(\tilde{\mu}_i - \tilde{\lambda}_i) \geq \varepsilon \sum f_i(\tilde{\lambda}) + \varepsilon.$$

Proof. Let P be the two-plane through $\tilde{\mu}$ spanned by $\nu_{\tilde{\mu}}$ and $\nu_{\tilde{\lambda}}$ (translated to $\tilde{\mu}$), and L the line on P through $\tilde{\mu}$ and perpendicular to $\nu_{\tilde{\lambda}}$. Since $0 < \nu_{\tilde{\mu}} \cdot \nu_{\tilde{\lambda}} \leq 1 - \beta^2/2$, L intersects $\partial B_\delta^\beta(\mu)$ at a unique point ζ , and therefore by the concavity of f ,

$$(2.18) \quad \sum f_i(\tilde{\lambda})(\tilde{\mu}_i - \tilde{\lambda}_i) = \sum f_i(\tilde{\lambda})(\zeta_i - \tilde{\lambda}_i) \geq f(\zeta) - f(\tilde{\lambda}) \geq \theta + f(\tilde{\mu}) - f(\tilde{\lambda}).$$

Next, we use the fact that $f(\tilde{\mu}) \geq f(\tilde{\lambda})$. Since $\zeta - \tilde{\mu}$ is perpendicular to $\nu_{\tilde{\lambda}}$ we see that

$$\nu_{\tilde{\lambda}} \cdot (\tilde{\mu} - \tilde{\lambda}) \geq \text{dist}(\zeta, \partial \Gamma^{f(\tilde{\lambda})}) \geq \text{dist}(\zeta, \partial \Gamma^{f(\tilde{\mu})}) \geq \text{dist}(\partial B_\delta^\beta(\tilde{\mu}), \partial \Gamma^{f(\tilde{\mu})}) \equiv \varepsilon_1 > 0.$$

It follows that

$$(2.19) \quad \sum f_i(\tilde{\lambda})(\tilde{\mu}_i - \tilde{\lambda}_i) \geq \frac{\varepsilon}{\sqrt{n}} \sum f_i(\tilde{\lambda}).$$

We obtain (2.17) for $\varepsilon = \min\{\theta/2, \varepsilon_1/2\sqrt{n}\}$. \square

Suppose now that $|\nu_{\tilde{\mu}} - \nu_{\tilde{\lambda}}| \geq \beta$. By Lemma 2.1 we may fix $a = O(b_1)$ sufficiently large in (2.14) to derive

$$(2.20) \quad \phi' F^{ii} U_{ii}^2 \leq C t a^2 \sum_{i \in J} F^{ii} + C t (a^2 + (\phi')^2 U_{11}^2) F^{11}$$

Note that

$$(2.21) \quad F^{ii} U_{ii}^2 \geq F^{11} U_{11}^2 + \sum_{i \in J} F^{ii} U_{ii}^2 \geq F^{11} U_{11}^2 + \frac{U_{11}^2}{9} \sum_{i \in J} F^{ii}.$$

Fixing γ sufficiently small (independent of $|\nabla u|_{C^0(\bar{M})}$) we obtain from (2.20) a bound $U_{11}(x_0) \leq C a / \sqrt{\gamma b} \leq C b_1^2$.

Suppose next $|\nu_{\tilde{\mu}} - \nu_{\tilde{\lambda}}| < \beta$. It follows that $\nu_{\tilde{\lambda}} - \beta \mathbf{1} \in \Gamma_n$ and therefore

$$(2.22) \quad F^{ii} \geq \frac{\beta}{\sqrt{n}} \sum F^{kk}, \quad \forall 1 \leq i \leq n.$$

Write $|\tilde{\lambda}|^2 = \sum \tilde{\lambda}_i^2 = \sum U_{ii}^2$. Since $\sum F^{ii}(\nabla_{ii} \underline{u} - \nabla_{ii} u) \geq 0$ by the concavity of f , we obtain from (2.14) and (2.22) (when γ is small enough),

$$(2.23) \quad \frac{\beta}{\sqrt{n}} |\tilde{\lambda}|^2 \sum F^{ii} \leq \sum F^{ii} \tilde{\lambda}_i^2 \leq C b_1^2 \left(1 + a^2 \sum F^{ii}\right).$$

By the concavity of f again,

$$\begin{aligned} |\tilde{\lambda}| \sum F^{ii} &\geq f(|\tilde{\lambda}| \mathbf{1}) - f(\tilde{\lambda}) + \sum F^{ii} \tilde{\lambda}_i \\ &\geq f(|\tilde{\lambda}| \mathbf{1}) - f(\tilde{\mu}) - \frac{1}{4|\tilde{\lambda}|} \sum F^{ii} \tilde{\lambda}_i^2 - |\tilde{\lambda}| \sum F^{ii}. \end{aligned}$$

Therefore,

$$(2.24) \quad |\tilde{\lambda}|^2 \sum F^{ii} \geq \frac{|\tilde{\lambda}|}{2} (f(|\tilde{\lambda}|\mathbf{1}) - f(\tilde{\mu})) - \frac{1}{8} \sum F^{ii} \tilde{\lambda}_i^2.$$

Suppose $|\tilde{\lambda}| \geq 1 + \max_{x \in \bar{M}} |\mu(x)| \equiv \Lambda$ and let

$$b_0 \equiv f(\Lambda\mathbf{1}) - \max_{x \in \bar{M}} f(\mu(x)) > 0.$$

We derive from (2.23) and (2.24) that

$$(2.25) \quad |\tilde{\lambda}|^2 \sum F^{ii} + b_0 |\tilde{\lambda}| \leq C b_1^2 \left(1 + a^2 \sum F^{ii}\right).$$

This gives a bound $|\tilde{\lambda}| \leq C b_1^2$. The proof of (1.12) is complete.

3. SECOND ORDER BOUNDARY ESTIMATES

In this section we first establish the boundary estimate (1.14) in Theorem 1.3. At the end of the section we remove condition (1.13) when M is a bounded domain in \mathbb{R}^n as needed in the proof of Theorem 1.1. We shall continue to use notations from the last section, and assume throughout the section that the function $\varphi \in C^\infty(\partial M)$ is extended smoothly to \bar{M} , still denoted φ .

For a point x_0 on ∂M , we shall choose smooth orthonormal local frames e_1, \dots, e_n around x_0 such that e_n is the interior normal to ∂M along the boundary. Let $\rho(x)$ and $d(x)$ denote the distances from $x \in \bar{M}$ to x_0 and ∂M , respectively. We may choose $\delta_0 > 0$ sufficiently small such that ρ and d are smooth in $M_{\delta_0} = \{x \in M : \rho(x) < \delta_0\}$. By a straightforward calculation (see also [4]),

$$(3.1) \quad |F^{ij} \nabla_{ij} \nabla_k (u - \varphi)| \leq C \left(1 + \sum f_i |\lambda_i| + \sum f_i\right), \quad \forall 1 \leq k \leq n.$$

The pure tangential second derivative estimates

$$(3.2) \quad |\nabla_{\alpha\beta} u(x_0)| \leq C, \quad \forall 1 \leq \alpha, \beta < n$$

directly follows from the boundary condition $u = \varphi$ on ∂M . To estimate the rest of second derivatives at x_0 we use the following barrier function as in [4],

$$(3.3) \quad \Psi = A_1 v + A_2 \rho^2 - A_3 \sum_{l < n} |\nabla_l (u - \varphi)|^2$$

where

$$(3.4) \quad v = (u - \underline{u}) + td - \frac{Nd^2}{2}.$$

The following lemma is key to our proof.

Lemma 3.1. *Assume that (1.3)-(1.5), (1.13) and (1.11) hold. Let $h \in C(\overline{M_{\delta_0}})$ satisfy $h \leq C\rho^2$ on $\overline{M_{\delta_0}} \cap \partial M$ and $h \leq C$ on $\overline{M_{\delta_0}}$. For any constant $K > 0$ there exist uniform positive constants t, δ sufficiently small, and A_1, A_2, A_3, N sufficiently large such that $\Psi \geq h$ on ∂M_δ and*

$$(3.5) \quad F^{ij} \nabla_{ij} \Psi \leq -K \left(1 + \sum f_i |\lambda_i| + \sum f_i \right) \text{ in } M_\delta.$$

Proof. For a point $x \in M_\delta$ we may assume that U_{ij} and F^{ij} are both diagonal at x . From (3.1) we see that

$$(3.6) \quad \sum_{l < n} F^{ij} \nabla_{ij} |\nabla_l(u - \varphi)|^2 \geq \sum_{l < n} F^{ij} U_{il} U_{jl} - C \left(1 + \sum f_i |\lambda_i| + \sum f_i \right).$$

By Proposition 2.19 in [4] there exists an index r such that

$$(3.7) \quad \sum_{l < n} F^{ij} U_{il} U_{jl} \geq \frac{1}{2} \sum_{i \neq r} f_i \lambda_i^2.$$

We consider two cases: **(a)** $|\nu_\mu - \nu_\lambda| < \beta$ and **(b)** $|\nu_\mu - \nu_\lambda| \geq \beta$ where $\mu = \lambda(\nabla^2 \underline{u} + \chi)$, $\lambda = \lambda(\nabla^2 u + \chi)$ and β is as in (2.15).

Case **(a)** $|\nu_\mu - \nu_\lambda| < \beta$. We first show that

$$(3.8) \quad \sum_{i \neq r} f_i \lambda_i^2 \geq c_0 \sum f_i \lambda_i^2 - C_0 \sum f_i$$

for some $c_0, C_0 > 0$. If $\lambda_r < 0$ this holds for $c_0 = \frac{1}{n+1}$, $C_0 = 0$ under condition (1.9) by Lemma 2.20 in [4]; however, we wish to prove it without using (1.9). Instead we shall use

$$(3.9) \quad f_i \geq \frac{\beta}{\sqrt{n}} \sum f_k, \quad \forall 1 \leq i \leq n$$

by (2.22) and the fact $\sum \lambda_i \geq 0$ which implies

$$(3.10) \quad \sum_{\lambda_i < 0} \lambda_i^2 \leq \left(- \sum_{\lambda_i < 0} \lambda_i \right)^2 \leq n \sum_{\lambda_i > 0} \lambda_i^2.$$

By (3.9) and (3.10),

$$f_r \lambda_r^2 \leq n f_r \sum_{\lambda_i > 0} \lambda_i^2 \leq \frac{n \sqrt{n}}{\beta} \sum_{\lambda_i > 0} f_i \lambda_i^2$$

if $\lambda_r < 0$. Suppose now that $\lambda_r > 0$. By the concavity of f ,

$$f_r \lambda_r \leq f_r \mu_r + \sum_{i \neq r} f_i (\mu_i - \lambda_i).$$

It follows from Schwarz inequality that

$$\begin{aligned} \frac{\beta f_r \lambda_r^2}{\sqrt{n}} \sum f_k &\leq f_r^2 \lambda_r^2 \leq 2f_r^2 \mu_r^2 + 2 \sum_{k \neq r} f_k \sum_{i \neq r} f_i (\mu_i - \lambda_i)^2 \\ &\leq \left(4 \sum_{i \neq r} f_i \lambda_i^2 + C \sum_{i \neq r} f_i \mu_i^2 \right) \sum f_k. \end{aligned}$$

This proves (3.8). It follow that

$$(3.11) \quad \frac{1}{4} \sum_{i \neq r} f_i \lambda_i^2 \geq K \sum f_i |\lambda_i| - \left(\frac{K^2}{c_0} + \frac{C_0}{2} \right) \sum f_i.$$

Suppose $|\lambda| \geq R$ for R sufficiently large. By (2.24) and (3.9) we obtain

$$\sum f_i \lambda_i^2 \geq b_0 |\lambda|$$

for some uniform $b_0 > 0$. In view of (3.6), (3.7) and (3.8) we can therefore choose $A_3 \geq 1$ such that

$$(3.12) \quad A_3 \sum_{l < n} F^{ij} \nabla_{ij} |\nabla_l(u - \varphi)|^2 \geq K \left(1 + \sum f_i |\lambda_i| \right) - C_1 \sum f_i.$$

for some $C_1 > 0$. From now on A_3 is fixed.

Note that

$$(3.13) \quad F^{ij} \nabla_{ij} v \leq C(t + Nd) \sum F^{ii} + F^{ij} \nabla_{ij}(u - \underline{u}) - N F^{ij} \nabla_i d \nabla_j d$$

and $F^{ij} \nabla_{ij}(u - \underline{u}) \leq 0$ by the concavity of f . Since $|\nabla d| \equiv 1$, by (3.9) we see that when N is sufficiently large,

$$(3.14) \quad F^{ij} \nabla_{ij} v \leq - \sum f_i \text{ in } M_\delta$$

for any $t \in (0, 1]$, as long as δ is sufficiently small. From (3.12) and (3.14) we derive

$$(3.15) \quad F^{ij} \nabla_{ij} \Psi \leq -K \left(1 + \sum f_i |\lambda_i| \right) + A_2 F^{ij} \nabla_{ij} \rho^2 + (C_1 - A_1) \sum f_i.$$

Suppose now that $|\lambda| \leq R$. There are uniform bounds (depending on R)

$$(3.16) \quad 0 < c_1 \leq \{F^{ij}\} \leq C_1$$

and therefore $F^{ij} \nabla_i d \nabla_j d \geq c_1$. Consequently by (3.13), for N sufficiently large,

$$(3.17) \quad F^{ij} \nabla_{ij} v \leq -1 \text{ in } M_\delta.$$

We now fix N such that (3.14) holds when $|\lambda| > R$ while (3.17) holds when $|\lambda| \leq R$.

Case **(b)** $|\nu_\mu - \nu_\lambda| \geq \beta$. By Lemma 2.1

$$F^{ij} \nabla_{ij}(\underline{u} - u) \geq \sum f_i (\mu_i - \lambda_i) \geq \varepsilon \left(1 + \sum f_i \right)$$

for some $\varepsilon > 0$. By (3.13) we may fix t, δ and then A_2 such that $v \geq 0$ on \bar{M}_δ ,

$$(3.18) \quad F^{ij} \nabla_{ij} v \leq -\frac{\varepsilon}{2} \left(1 + \sum f_i\right) \text{ in } M_\delta$$

and

$$A_2 \rho^2 \geq h + A_3 \sum_{l < n} |\nabla_l(u - \varphi)|^2 \text{ on } \partial M_\delta.$$

Finally, we can choose A_1 sufficiently large so that (3.5) holds. In case **(a)** (3.5) which holds without assumption (1.13) follows from (3.15) when $|\lambda| > R$, and from (3.16) and (3.17) when $|\lambda| \leq R$. For case **(b)**, we obtain (3.5) from (3.7), (3.18) and the following inequality which is a consequence of (1.3), (1.4) and (1.13):

$$(3.19) \quad \sum f_i |\lambda_i| \leq \epsilon \sum_{i \neq r} f_i \lambda_i^2 + \frac{C}{\epsilon} \sum f_i + C$$

for any $\epsilon > 0$ and index r . Indeed, if $\lambda_r < 0$ then

$$\sum f_i |\lambda_i| = 2 \sum_{\lambda_i > 0} f_i \lambda_i - \sum f_i \lambda_i \leq \epsilon \sum_{\lambda_i > 0} f_i \lambda_i^2 + \frac{1}{\epsilon} \sum_{\lambda_i > 0} f_i + K_0 \sum f_i + K_0$$

by assumption (1.13). Similarly, if $\lambda_r > 0$ then by the concavity of f ,

$$\begin{aligned} \sum f_i |\lambda_i| &= \sum f_i \lambda_i - 2 \sum_{\lambda_i < 0} f_i \lambda_i \\ &\leq \epsilon \sum_{\lambda_i < 0} f_i \lambda_i^2 + \frac{1}{\epsilon} \sum_{\lambda_i < 0} f_i + \sum f_i \mu_i + f(\lambda) - f(\mu) \end{aligned}$$

where $\mu = \lambda(\nabla^2 \underline{u} + \chi)$. So we have (3.19). \square

A bound for the mixed tangential-normal derivatives

$$(3.20) \quad |\nabla_{n\alpha} u(x_0)| \leq C, \quad \forall \alpha < n$$

follows immediately from (3.1) and (3.5) in Lemma 3.1, while the double normal derivative estimate

$$(3.21) \quad |\nabla_{nn} u(x_0)| \leq C$$

can be derived as in [4] using Lemma 3.1 in place of Lemma 4.1 in [4]. We outline the proof here for completeness, and refer the reader to [4] for more details.

For a $(0, 2)$ tensor W on \bar{M} and $x \in \partial M$, let $\tilde{W}(x)$ denote the restriction of W to $T_x \partial M$ and $\lambda'(\tilde{W})$ the eigenvalue of \tilde{W} (with respect to the induced metric). It suffices to show that there are uniform constants $c_0, R_0 > 0$ such that $(\lambda'(\tilde{U}(x)), R) \in \Gamma$ and

$$(3.22) \quad f(\lambda'(\tilde{U}(x)), R) \geq \psi(x) + c_0$$

for all $R > R_0$ and $x \in \partial M$. Following an idea of Trudinger [9] we consider

$$\tilde{m} \equiv \liminf_{R \rightarrow \infty} \min_{x \in \partial M} [f(\lambda'(\tilde{U}(x)), R) - \psi(x)]$$

and assume $\tilde{m} < \infty$ (otherwise we are done).

Suppose that \tilde{m} is achieved at a point $x_0 \in \partial M$ and choose local orthonormal frames (e_1, \dots, e_n) around x_0 as before such that $U_{\alpha\beta}(x_0)$ ($1 \leq \alpha, \beta \leq n-1$) is diagonal. Define for a symmetric $(n-1)^2$ matrix $\{r_{\alpha\beta}\}$ with $(\lambda'[\{r_{\alpha\beta}\}], R) \in \Gamma$ for R sufficiently large,

$$\tilde{F}[r_{\alpha\beta}] = \lim_{R \rightarrow +\infty} f(\lambda'[\{r_{\alpha\beta}\}], R),$$

where $\lambda'[\{r_{\alpha\beta}\}]$ denotes the eigenvalues of the matrix $\{r_{\alpha\beta}\}$ ($1 \leq \alpha, \beta \leq n-1$). Note that \tilde{F} is finite and concave since f is concave and continuous.

By the concavity of \tilde{F} there is a symmetric matrix $\{\tilde{F}_0^{\alpha\beta}\}$ such that

$$(3.23) \quad \tilde{F}_0^{\alpha\beta}(r_{\alpha\beta} - U_{\alpha\beta}(x_0)) \geq \tilde{F}[r_{\alpha\beta}] - \tilde{F}[U_{\alpha\beta}(x_0)]$$

for any symmetric matrix $\{r_{\alpha\beta}\}$ with $(\lambda'[\{r_{\alpha\beta}\}], R) \in \Gamma$; $\{\tilde{F}_0^{\alpha\beta}\}$ is uniquely defined if and only if \tilde{F} is differentiable at $U_{\alpha\beta}(x_0)$ in which case

$$\tilde{F}_0^{\alpha\beta} = \frac{\partial \tilde{F}}{\partial r_{\alpha\beta}}[U_{\alpha\beta}(x_0)].$$

Let $\sigma_{\alpha\beta} = \langle \nabla_{\alpha} e_{\beta}, e_n \rangle$. Since $u - \underline{u} = 0$ on ∂M ,

$$(3.24) \quad U_{\alpha\beta} - \underline{U}_{\alpha\beta} = -\nabla_n(u - \underline{u})\sigma_{\alpha\beta} \quad \text{on } \partial M.$$

It follows that

$$\nabla_n(u - \underline{u})\tilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta}(x_0) \geq \tilde{F}[\underline{U}_{\alpha\beta}(x_0)] - \tilde{F}[U_{\alpha\beta}(x_0)] \geq \tilde{c} - \tilde{m}$$

where

$$\tilde{c} \equiv \liminf_{R \rightarrow \infty} \min_{x \in \partial M} [f(\lambda'(\tilde{U}(x)), R) - f(\lambda(\tilde{U}(x)))] > 0.$$

Consequently, if

$$\nabla_n(u - \underline{u})(x_0)\tilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta}(x_0) \leq \tilde{c}/2$$

then $\tilde{m} \geq \tilde{c}/2 > 0$ and we are done.

Suppose now that

$$\nabla_n(u - \underline{u})(x_0)\tilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta}(x_0) > \frac{\tilde{c}}{2}$$

and let $\eta \equiv \tilde{F}_0^{\alpha\beta}\sigma_{\alpha\beta}$. Note that

$$(3.25) \quad \eta(x_0) \geq \tilde{c}/2 \nabla_n(u - \underline{u})(x_0) \geq 2\epsilon_1 \tilde{c}$$

for some uniform $\epsilon_1 > 0$. We may assume $\eta \geq \epsilon_1 \tilde{c}$ on \bar{M}_{δ} by requiring δ small. Define in M_{δ} ,

$$\begin{aligned} \Phi &= -\nabla_n(u - \varphi) + \frac{1}{\eta} \tilde{F}_0^{\alpha\beta} (\nabla_{\alpha\beta} \varphi + \chi_{\alpha\beta} - U_{\alpha\beta}(x_0)) - \frac{\psi - \psi(x_0)}{\eta} \\ &\equiv -\nabla_n(u - \varphi) + Q. \end{aligned}$$

We have $\Phi(x_0) = 0$ and $\Phi \geq 0$ on ∂M near x_0 , and by (3.1),

$$(3.26) \quad F^{ij} \nabla_{ij} \Phi \leq -F^{ij} \nabla_{ij} \nabla_n u + C \sum F^{ii} \leq C \left(1 + \sum f_i |\lambda_i| + \sum f_i \right).$$

By (3.5) in Lemma 3.1, for $A_1 \gg A_2 \gg A_3 \gg 1$ we derive $\Psi + \Phi \geq 0$ on ∂M_δ and

$$(3.27) \quad F^{ij} \nabla_{ij} (\Psi + \Phi) \leq 0 \text{ in } M_\delta.$$

By the maximum principle, $\Psi + \Phi \geq 0$ in M_δ and therefore $\Phi_n(x_0) \geq -\nabla_n \Psi(x_0) \geq -C$. This gives $\nabla_{nn} u(x_0) \leq C$.

So we have an *a priori* upper bound for all eigenvalues of $\{U_{ij}(x_0)\}$. Consequently, $\lambda[\{U_{ij}(x_0)\}]$ is contained in a compact subset of Γ by (1.5). It follows that

$$\tilde{m} \geq m_R \equiv F([U_{\alpha\beta}(x_0)], R) - \psi(x_0) > 0$$

when R is sufficiently large. The proof of (1.14) in Theorem 1.3 is therefore complete.

We finish this section by deriving (1.14) without assumption (1.13) when $M = \Omega$ is a bounded smooth domain in \mathbb{R}^n , which is needed in the proof of Theorem 1.1. For this purpose we make use of a formula from [1].

Consider an arbitrary point on $\partial\Omega$, which we may assume to be the origin of \mathbb{R}^n with the positive x_n axis in the interior normal direction to $\partial\Omega$ at the origin. There exists a uniform constant $r > 0$ such that $\partial\Omega \cap B_r(0)$ can be represented as a graph

$$(3.28) \quad x_n = \frac{1}{2} \sum_{\alpha, \beta < n} B_{\alpha\beta} x_\alpha x_\beta + O(|x'|^3), \quad x' = (x_1, \dots, x_{n-1}).$$

Let

$$(3.29) \quad T_\alpha = \partial_\alpha + \sum_{\beta < n} B_{\alpha\beta} (x_\beta \partial_n - x_n \partial_\beta)$$

for $\alpha < n$ and $T_n = \partial_n$. From [1] we have

$$(3.30) \quad F^{ij} (T_\alpha u)_{ij} = T_\alpha \psi.$$

Therefore, since $u = \varphi$ on $\partial\Omega$,

$$(3.31) \quad |F^{ij} (T_\alpha (u - \varphi))_{ij}| \leq C \sum F^{ii} + C \text{ in } \Omega_r \equiv \Omega \cap B_r(0), \quad 1 \leq \alpha \leq n,$$

and

$$(3.32) \quad |T_\alpha (u - \varphi)| \leq C |x|^2 \text{ on } \partial\Omega \cap B_r(0), \quad 1 \leq \alpha \leq n-1.$$

In order to derive (1.14) without assumption (1.13) we consider in place (3.3) the barrier function

$$(3.33) \quad \Psi = A_1 v + A_2 |x|^2 - A_3 \sum_{l < n} |T_l (u - \varphi)|^2$$

where v is given in (3.4). By (3.31) we see that (compare with (3.6))

$$(3.34) \quad \sum_{l < n} F^{ij} (|T_l (u - \varphi)|^2)_{ij} \geq \sum_{l < n} F^{ij} (T_l u)_i (T_l u)_j - C \sum F^{ii} - C;$$

So we have the following modification of Lemma 3.1.

Lemma 3.2. *Let $h \in C(\overline{\Omega_{\delta_0}})$ satisfy $h \leq C|x|^2$ on $\overline{\Omega_{\delta_0}} \cap \partial\Omega$ and $h \leq C$ on $\overline{\Omega_{\delta_0}}$. Under assumptions (1.3)-(1.5) and (1.11), for any constant $K > 0$ there exist uniform positive constants t, δ sufficiently small, and A_1, A_2, A_3, N sufficiently large such that $\Psi \geq h$ on $\partial\Omega_\delta$ and*

$$(3.35) \quad F^{ij} \nabla_{ij} \Psi \leq -K \sum F^{ii} - K \quad \text{in } \Omega_\delta.$$

Using Lemma 3.2 in place of Lemma 3.1 in the above proof we can derive (1.14) without assumption (1.13); we leave the details to the reader.

4. GRADIENT ESTIMATES AND EXISTENCE

In order to prove Theorem 1.5 using the continuity method we need the gradient estimate (1.19). We shall only consider cases (ii) and (iv) as the other two cases are already known; see [4].

Suppose $|\nabla u|e^\phi$ achieves a maximum at an interior point $x_0 \in M$. As in Section 2 we choose orthonormal local frames at x_0 such that both U_{ij} and F^{ij} are both diagonal at x_0 . Then at x_0 ,

$$\begin{aligned} \frac{\nabla_k u \nabla_{ik} u}{|\nabla u|^2} + \nabla_i \phi &= 0, \\ F^{ii} \frac{\nabla_k u \nabla_{iik} u + \nabla_{ik} u \nabla_{ik} u}{|\nabla u|^2} - 2F^{ii} \frac{(\nabla_k u \nabla_{ik} u)^2}{|\nabla u|^4} + F^{ii} \nabla_{ii} \phi &\leq 0. \end{aligned}$$

For any $0 < \epsilon < 1$,

$$\sum_k (\nabla_{ik} u)^2 = \sum_k (U_{ik} - \chi_{ik})^2 \geq (1 - \epsilon) U_{ii}^2 - \frac{C}{\epsilon}.$$

Similarly,

$$\left(\sum_k \nabla_k u \nabla_{ik} u \right)^2 \leq (1 + \epsilon) |\nabla_i u|^2 U_{ii}^2 + \frac{C}{\epsilon} |\nabla u|^2.$$

Let $J = \{i : (n+2)|\nabla_i u|^2 > |\nabla u|^2\}$, $K_1 = \inf_{k,l} R_{klkl}$ and $\epsilon = \frac{1}{2n+3}$ so that

$$\frac{n(1-\epsilon)}{n+1} = \frac{2n}{2n+3} \geq \frac{4}{2n+3} = \frac{2(1+\epsilon)}{n+2}.$$

We derive

$$\begin{aligned} (4.1) \quad & \frac{1-\epsilon}{n+1} F^{ii} U_{ii}^2 - 2|\nabla u|^2 \sum_{i \in J} F^{ii} |\nabla_i \phi|^2 \\ & + |\nabla u|^2 F^{ii} \nabla_{ii} \phi \leq C(1 - K_1 |\nabla u|^2) \sum F^{ii} + C|\nabla u|. \end{aligned}$$

Let $v = \underline{u} - u + \inf_{\bar{M}}(u - \underline{u}) + 1$ and

$$\phi = -\log(1 - bv^2) + A(\underline{u} + w - \inf_{\bar{M}}(\underline{u} + w))$$

where $b = 1/2 \max v^2$ and A is a constant to be determined; in case (ii) where $K_1 \geq 0$ we shall take $A = 0$. We have

$$\nabla_i \phi = \frac{2bv \nabla_i v}{1 - bv^2} + A \nabla_i(\underline{u} + w)$$

and

$$\nabla_{ii} \phi = \frac{2bv \nabla_{ii} v + 2b |\nabla_i v|^2}{1 - bv^2} + \frac{4b^2 v^2 |\nabla_i v|^2}{(1 - bv^2)^2} + A \nabla_{ii}(\underline{u} + w).$$

By (4.1),

$$(4.2) \quad \begin{aligned} & \frac{1 - \epsilon}{n + 1} F^{ii} U_{ii}^2 + |\nabla u|^2 \sum_{i \in J} F^{ii} (b |\nabla_i v|^2 - C A^2) + \frac{2bv |\nabla u|^2}{1 - bv^2} F^{ii} \nabla_{ii} v \\ & + A |\nabla u|^2 F^{ii} \nabla_{ii}(\underline{u} + w) \leq C(1 - K_1 |\nabla u|^2) \sum F^{ii} + C |\nabla u|. \end{aligned}$$

Let $\mu = \lambda(\nabla^2 \underline{u}(x_0) + \chi(x_0))$, $\lambda = \lambda(\nabla^2 u(x_0) + \chi(x_0))$ and β as in (2.15). Suppose first that $|\nu_\mu - \nu_\lambda| \geq \beta$. By Lemma 2.1,

$$F^{ii} \nabla_{ii} v = F^{ii} \nabla_{ii}(\underline{u} - u) \geq \varepsilon \sum F^{ii} + \varepsilon$$

for some $\varepsilon > 0$. Let $A = A_1 K_1^- / \varepsilon$, $K_1^- = \max\{-K_1, 0\}$ and fix A_1 sufficiently large. A bound $|\nabla u| \leq C$ follows from (4.2) in both case (ii) and case (iv).

We now consider the case $|\nu_\mu - \nu_\lambda| < \beta$. From (4.2) we see that if $|\nabla u|$ is sufficiently large,

$$(4.3) \quad F^{ii} U_{ii}^2 + c_1 |\nabla u|^4 \sum_{i \in J} F^{ii} \leq C(1 - K_1 |\nabla u|^2) \sum F^{ii} + C |\nabla u|$$

where $c_1 > 0$. Note that $J \neq \emptyset$.

Suppose $|\lambda| \geq R$ for R sufficiently large. By (2.22) and (2.24) we obtain

$$(4.4) \quad \begin{aligned} F^{ii} U_{ii}^2 + c_1 |\nabla u|^4 \sum_{i \in J} F^{ii} & \geq \frac{\beta}{\sqrt{n}} (n |\lambda|^2 + c_1 |\nabla u|^4) \sum F^{ii} \\ & \geq 2\beta |\lambda| \sqrt{c_1} |\nabla u|^2 \sum F^{ii} \\ & \geq c_2 |\nabla u|^2 \end{aligned}$$

for some uniform $c_2 > 0$. Choosing a larger R if necessary, we obtain from (4.3) and (4.4) a bound for $|\nabla u(x_0)|$.

Suppose $|\lambda| \leq R$. By (1.3) and (1.5) there exists $C_1 > 0$ depending on R such that

$$C_1^{-1} \leq \{F^{ii}\} \leq C_1.$$

We derive a bound for $|\nabla u(x_0)|$ from (4.3) again. The proof of (1.19) is thus complete.

Finally, by the maximum principle we have $\underline{u} \leq u \leq h$ where $h \in C^2(\bar{M})$ is the solution of $\Delta h + \text{tr} \chi = 0$ in \bar{M} with $h = \varphi$ on ∂M . This gives bounds for $|u|_{C^0(\bar{M})}$ and $|\nabla u|$ on ∂M . We thus have derived a bound $|u|_{C^2(\bar{M})} \leq C$ for each case in Theorem 1.5. By Evans-Krylov theorem we obtain $|u|_{C^{2,\alpha}(\bar{M})} \leq C$. Higher order estimates now follow from the Schauder theory for linear uniformly elliptic equations, and Theorem 1.5 can be proved using the continuity method; we omit the proof here as it is standard and well known.

5. PROOF OF THEOREM 1.4

We first modify the definition of the *tangent cone at infinity* for the function f when (1.4) is replaced by (1.17).

Consider a function $Y \in C^2(\mathbb{R}^{n-1})$ which is bounded from below and convex outside a ball $B_{R_0}(0)$, i.e.

$$Y_{x_i x_j}(x) \xi_i \xi_j \geq 0 \quad \forall \xi \in \mathbb{R}^{n-1}, \quad \text{in } \mathbb{R}^{n-1} \setminus B_{R_0}(0).$$

Let Σ denote the graph of Y and L_x the function whose graph is the tangent plane of Σ at $(x, Y(x)) \in \mathbb{R}^n$. For $R \geq R_0$ define

$$\mathcal{C}_\Sigma(R) = \{(y, y_n) \in \mathbb{R}^n : |y| < R, y_n > L_x(y) \forall x \in \partial B_R(0)\}.$$

Clearly $\mathcal{C}_\Sigma(R)$ is open and $\mathcal{C}_\Sigma(R) \subset \mathcal{C}_\Sigma(R')$ for $R' \geq R \geq R_0$. Therefore,

$$\mathcal{C}_\Sigma := \lim_{R \rightarrow +\infty} \mathcal{C}_\Sigma(R)$$

is well defined and open.

Recall from the Introduction $\Gamma^\sigma = \{\lambda \in \Gamma : f(\lambda) > \sigma\}$. By (1.3) we see that $\partial\Gamma^\sigma = \{\lambda \in \Gamma : f(\lambda) = \sigma\}$ is a smooth graph in \mathbb{R}^n over the plane perpendicular to $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$. So $\mathcal{C}_{\partial\Gamma^\sigma}$ is well defined if (1.17) holds. It is easy to see that $\mathcal{C}_{\partial\Gamma^\sigma}$ coincides with \mathcal{C}_σ^+ defined in [4] when (1.4) holds; from now on we shall still use the notation $\mathcal{C}_\sigma^+ := \mathcal{C}_{\partial\Gamma^\sigma}$ in the general case. We have the following extension of Theorem 2.17 in [4].

Lemma 5.1. *Suppose that f satisfies (1.3) and (1.17). Let $[a, b] \subset (\sup_{\partial\Gamma} f, \sup_\Gamma f)$ and for each $\sigma \in [a, b]$, $K_\sigma = \overline{\Gamma_\sigma \cap B_{r_0}(0)}$ where $r_0 > 0$. There exist constants $\theta > 0$, $R > R_0$ such that for all $\sigma \in [a, b]$,*

$$(5.1) \quad \sum f_i(\lambda)(\mu_i - \lambda_i) \geq \theta \sum f_i(\lambda) + \theta, \quad \forall \mu \in K_\sigma, \quad \forall \lambda \in \partial\Gamma^\sigma \setminus B_R(0).$$

Proof. Let $d_\sigma(\mu) = \text{dist}(\mu, \partial\mathcal{C}_\sigma^+)$ be the distance from μ to $\partial\mathcal{C}_\sigma^+$; $d_\sigma(\mu)$ is continuous in both μ and σ since the hypersurfaces $\{\partial\Gamma^\sigma : \sigma \in [a, b]\}$ form a smooth foliation of the region bounded by $\partial\Gamma^a$ and $\partial\Gamma^b$. As each K_σ is compact in \mathcal{C}_σ^+ which is open,

$$\bar{d} := \min_{a \leq \sigma \leq b} \min_{\mu \in K_\sigma} d_\sigma(\mu) > 0.$$

Consequently, for each σ there exists $R_\sigma \geq R_0$ such that

$$\frac{\bar{d}}{2} \leq d_{\sigma,R} := \min_{\lambda \in \partial\Gamma^\sigma \cap \partial B_R} \min_{\mu \in K_\sigma} \text{dist}(\mu, T_\lambda \partial\Gamma^\sigma), \quad \forall R \geq R_\sigma.$$

Following [4] one can show that $d_{\sigma,R}$ is non-decreasing in R and

$$\bar{R} := \max_{a \leq \sigma \leq b} R_\sigma < +\infty$$

if we redefine $R_\sigma := \min\{R \geq R_0 : \bar{d} \leq 2d_{\sigma,R}\}$. This implies

$$(5.2) \quad \sum f_i(\lambda)(\mu_i - \lambda_i) \geq \frac{\bar{d}}{2\sqrt{n}} \sum f_i(\lambda), \quad \forall \mu \in K_\sigma, \quad \forall \lambda \in \partial\Gamma^\sigma \setminus B_{\bar{R}}(0).$$

Without loss of generality we may assume $r_0 \leq \bar{R}$. Let $\mu \in K_\sigma$ and $\lambda \in \partial\Gamma^\sigma$ with $|\lambda| \geq 7\bar{R}$. There is a unique point $\zeta = t\mu + (1-t)\lambda \in \partial B_{2\bar{R}}(0)$. We have $|\mu - \zeta| \leq 3\bar{R}$ and $|\mu - \lambda| \geq 6\bar{R}$.

Let $\eta \in \partial\Gamma^\sigma$ such that $|\zeta - \eta| = \text{dist}(\zeta, \partial\Gamma^\sigma)$. Since μ and λ are on the same side of $T_\eta \partial\Gamma^\sigma$,

$$|\zeta - \eta| = (\zeta - \eta) \cdot \nu_\eta \geq \frac{1}{2}(\mu - \eta) \cdot \nu_\eta \geq \frac{\bar{d}}{4}.$$

Therefore $f(\zeta) \geq \sigma + c_0$ for some uniform $c_0 > 0$ (independent of σ). By the concavity of f in $\Gamma \setminus B_{r_0}(0)$,

$$(5.3) \quad \sum f_i(\lambda)(\mu_i - \lambda_i) \geq \sum f_i(\lambda)(\zeta_i - \lambda_i) \geq f(\zeta) - f(\lambda) \geq c_0.$$

Combining (5.2) and (5.3) completes the proof. \square

We can now modify the proof of Theorem 1.3 to show Theorem 1.4 in an obvious way. In stead of the cases (a) $|\nu_\lambda - \nu_\mu| < \beta$ and (b) $|\nu_\lambda - \nu_\mu| \geq \beta$, we consider separately $|\lambda| > R$ which corresponds to (b), or $|\lambda| \leq R$. For (1.12), we use Lemma 5.1 when $|\lambda| \geq R$ and choose a sufficiently large in (2.14) to derive (2.20). Similarly we use Lemma 5.1 when $|\lambda| \geq R$ to replace Lemma 2.1 in the proof of the gradient estimate (1.15), and in that of Lemma 3.1 for the second order boundary estimate (1.14).

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